

Optimal Solution to a Transformation Problem of a Triangular Array

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Abstract

We find the general solution to an optimization problem by applying basic knowledge of algebra, geometry, and vectors. The problem consists in transforming an equilateral triangular array of coins, with n coins on each side, into a similar triangle in an upside down position using the least number of moves. The answer, $\lfloor n(n+1)/6 \rfloor$, is classical; the proof given here reduces the problem to finding the closest integer point to a fixed point on the plane $k_1 + k_2 + k_3 = n$, and also produces every optimal configuration. We then treat the same problem in any dimension: for a d -dimensional pile of spheres we prove that an optimal placement of the inverted pile is always balanced and identify it exactly, obtain the closed form for tetrahedral piles, and report that the three- and two-dimensional pattern of disjoint corner pieces breaks down from dimension four on.

1 Introduction

In this paper we will find the general solution to an optimization problem by applying basic knowledge of algebra, geometry, and vectors. The problem consists in transforming an equilateral triangular array of coins to a similar triangle in an upside down position. This has to be accomplished in the least amount of moves possible.

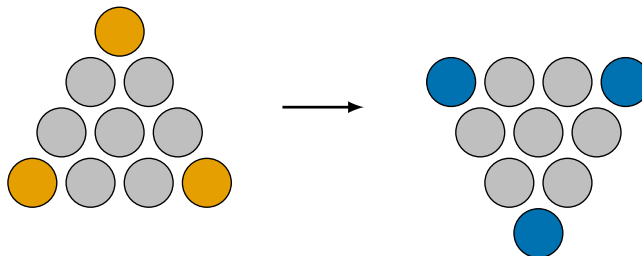


Figure 1: Transformation. The three orange coins on the left are moved to the three blue positions on the right.

An example could be the case from Figure 1, in which the triangle has a base of 4 coins, followed by 3 on the next row, then 2 and 1 at the very top. In order to transform the triangular array, 3 coins would be the least amount of coins that need to be moved.

The problem is an old one. Martin Gardner posed it in his February 1966 *Scientific American* column and reported the answer, divide the number of coins by three and ignore the remainder, without proof [2]. The answer appears again in the mathematics-education literature [1], in puzzle collections [3], as sequence A001840 in the On-Line Encyclopedia of Integer Sequences

[5], and in a 2018 preprint whose authors wrote that they could not fully explain why the division rule works [4]. The existing proofs balance three corner triangles or use a recursion in n . The solution below is different in character: after reducing the problem to three integers, we complete the square and turn the minimization into a nearest-point problem on a plane, which we solve by orthogonal projection.

2 Notation

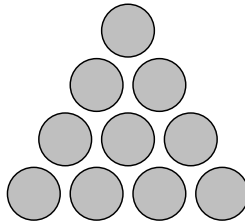


Figure 2: Triangle created by coins.

In this problem we create a triangular shaped array of coins, as shown in Figure 2. Each row is composed of an increasing number of coins, starting with one coin, and on each row, a coin on the row above it is placed “between” two coins in that row. This formation creates a shape that resembles an equilateral triangle. In Figure 2 we have four coins in the base of the triangle; in our problem we will denote the number of coins in the base with a lower case n , meaning any number of coins, and the total number of coins in the figure is represented by T , so that

$$T = \frac{n(n+1)}{2}.$$

Our approach to solve this problem involves minimizing the number of coins that have to be removed from the top and the bottom sides; this number is denoted by t . The coins remaining from what we had have to create a figure that is symmetric with respect to a horizontal axis that divides the figure exactly in two equal parts. Those coins that were removed from the top are now placed on the bottom, and those removed from the bottom sides are now placed on the top sides. In Figure 1 we can see what we explained in the previous sentences. In Figure 3 we can see how $k_1 + k_2 + k_3 = n$, where k_1 , k_2 and k_3 represent numbers of coins that add up to n .

To make the counting precise we place the coins on the triangular lattice and use integer coordinates: the coin $(a, b) \in \mathbb{Z}^2$ sits at the point $(a + \frac{b}{2}, \frac{\sqrt{3}}{2}b)$ of the plane. The original triangle is

$$U = \{(a, b) \in \mathbb{Z}^2 : a \geq 0, b \geq 0, a + b \leq n - 1\},$$

with base row $b = 0$ and apex $(0, n - 1)$. Rotating U by 180° and translating by $(p, q) \in \mathbb{Z}^2$ gives the upside down triangle

$$D(p, q) = \{(a, b) \in \mathbb{Z}^2 : a \leq p, b \leq q, a + b \geq p + q - (n - 1)\},$$

whose horizontal edge is the row $b = q$ and whose apex is $(p, q - n + 1)$. Every upside down triangle on the lattice is of this form. A *move* takes one coin from its position and places it on an empty position of the lattice. The question is the least number of moves that takes U to some $D(p, q)$.

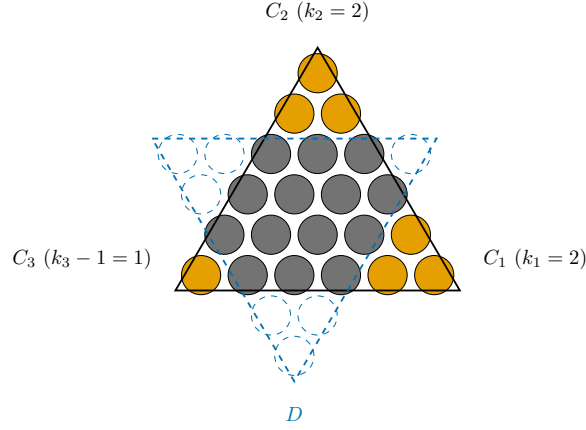


Figure 3: Triangle with $k_1 + k_2 + k_3 = n$, shown for $n = 6$. The original triangle has the solid outline and the inverted one the dashed outline. Gray coins stay in place; the orange coins in the three corner triangles C_1 , C_2 , C_3 must be moved to the dashed positions.

3 Solution

In our process of solving this problem we will encounter different theorems which will help us to arrive at one general formula for any amount of coins given. Our first theorem will be of great importance in order to understand the problem.

Theorem 1. *Place the upside down triangle over the original one in any way that leaves at least one coin in place, as in Figure 3. Then the coins that have to be moved form three triangles with bases k_1 , k_2 and $k_3 - 1$, where k_1, k_2, k_3 are nonnegative integers with*

$$k_1 + k_2 + k_3 = n,$$

and the number of coins that have to be moved is

$$t = \frac{k_1(k_1 + 1)}{2} + \frac{k_2(k_2 + 1)}{2} + \frac{k_3(k_3 - 1)}{2}. \quad (1)$$

Conversely, every such triple (k_1, k_2, k_3) comes from some placement of the upside down triangle.

Proof. First, for a fixed placement $D = D(p, q)$, the number of moves needed is exactly the number of coins of U not covered by D , that is $|U \setminus D|$: each such coin must move at least once, and since $|U| = |D|$ there are exactly as many empty positions in $D \setminus U$ as coins in $U \setminus D$, so moving each uncovered coin once suffices.

A coin (a, b) of U is not covered by D when it fails one of the three inequalities defining D , so

$$U \setminus D = C_1 \cup C_2 \cup C_3, \quad C_1 = \{a \geq p + 1\}, \quad C_2 = \{b \geq q + 1\}, \quad C_3 = \{a + b \leq p + q - n\},$$

each intersected with U . These are the three corner triangles in Figure 3. Set

$$k_1 = n - 1 - p, \quad k_2 = n - 1 - q, \quad k_3 = p + q - n + 2,$$

so that $k_1 + k_2 + k_3 = n$ automatically.

We may assume $0 \leq p \leq n-1$, $0 \leq q \leq n-1$ and $p+q \geq n-2$; then $k_1, k_2, k_3 \geq 0$. Indeed, if $p < 0$ or $q < 0$ or $p+q \geq 2n-1$ then D covers no coin of U at all. If $p \geq n$, the condition $a \leq p$ is automatic on U and $U \cap D(p, q) \subseteq U \cap D(n-1, q)$ because $p+q-n+1 \geq q+1$, so lowering p to $n-1$ covers at least as many coins; similarly for q . If $p+q \leq n-3$, then $U \cap D(p+1, q) \supseteq U \cap D(p, q)$ because the third inequality is vacuous on U in both cases, so raising p covers at least as many coins. None of these changes increases the number of moves, so for the purpose of minimizing t the assumption costs nothing.

Under this assumption the three corners are disjoint: a coin in $C_1 \cap C_2$ would need $p+q+2 \leq a+b \leq n-1$, that is $p+q \leq n-3$; a coin in $C_1 \cap C_3$ would need $p+1 \leq a \leq a+b \leq p+q-n$, that is $q \geq n+1$; and symmetrically for $C_2 \cap C_3$. Writing $a = a' + p + 1$ shows that $C_1 = \{a', b \geq 0, a' + b \leq k_1 - 1\}$ is a triangle with base k_1 , so $|C_1| = k_1(k_1 + 1)/2$; in the same way $|C_2| = k_2(k_2 + 1)/2$; and $C_3 = \{a, b \geq 0, a + b \leq k_3 - 2\}$ is a triangle with base $k_3 - 1$, so $|C_3| = (k_3 - 1)k_3/2$ (zero when k_3 is 0 or 1). Adding gives (1).

Conversely, given nonnegative k_1, k_2, k_3 with sum n , the placement $(p, q) = (n-1-k_1, n-1-k_2)$ satisfies $p+q = n-2+k_3 \geq n-2$ and produces these three corners. $\square$

We now know that from Figure 3 the triangle created by the base k_1 has $\frac{k_1(k_1+1)}{2}$ coins, the triangle created by the base k_2 has $\frac{k_2(k_2+1)}{2}$ coins, and the triangle created by the base $k_3 - 1$ has $\frac{k_3(k_3-1)}{2}$ coins. By symmetry of the figure, the three empty triangles of the upside down position have the same sizes, which is why the moved coins fit exactly.

Theorem 2. *Referring back to Theorem 1 and letting t be the number of coins that need to be moved, then*

$$2t = \left(k_1 + \frac{1}{2}\right)^2 + \left(k_2 + \frac{1}{2}\right)^2 + \left(k_3 - \frac{1}{2}\right)^2 - \frac{3}{4}.$$

Proof. Let us begin by stating that

$$t = \frac{k_1(k_1+1)}{2} + \frac{k_2(k_2+1)}{2} + \frac{k_3(k_3-1)}{2}.$$

Multiplying both sides by 2 we get

$$2t = k_1(k_1+1) + k_2(k_2+1) + k_3(k_3-1),$$

or also

$$2t = k_1^2 + k_1 + k_2^2 + k_2 + k_3^2 - k_3.$$

By completing the square we have

$$2t = \left(k_1 + \frac{1}{2}\right)^2 + \left(k_2 + \frac{1}{2}\right)^2 + \left(k_3 - \frac{1}{2}\right)^2 - \frac{3}{4}. \quad \square$$

From the second theorem we can see that if we just minimize the part

$$S = \left(k_1 + \frac{1}{2}\right)^2 + \left(k_2 + \frac{1}{2}\right)^2 + \left(k_3 - \frac{1}{2}\right)^2$$

we will minimize the total amount of coins moved, since $2t = S - \frac{3}{4}$. Now S is the square of the distance from the point (k_1, k_2, k_3) to the fixed point

$$P = \left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right),$$

and the point (k_1, k_2, k_3) is restricted to the plane $k_1 + k_2 + k_3 = n$. To accomplish the minimization we will be applying the following theorem.

Theorem 3. Let Q be a point located on the plane $k_1 + k_2 + k_3 = n$. If Q is the closest point of the plane to the point $P = (-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$ then

$$Q = \left(\frac{n-1}{3}, \frac{n-1}{3}, \frac{n+2}{3} \right).$$

Proof. Let $k_1 + k_2 + k_3 = n$ be the equation of the plane with normal vector $(1, 1, 1)$. Now the perpendicular line from the plane to the point $(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$ has equation

$$d(l) = \left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2} \right) + l(1, 1, 1) = \left(l - \frac{1}{2}, l - \frac{1}{2}, l + \frac{1}{2} \right).$$

The point of intersection between the line and the plane, which is denoted by $Q = (k_1, k_2, k_3)$, is the closest point to $(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$, satisfying

$$k_1 = k_2 \quad \text{and} \quad k_3 = k_1 + 1.$$

From $d(l)$ the following is true: $k_1 = l - \frac{1}{2}$ and $k_2 = l - \frac{1}{2}$, therefore $k_1 = k_2$. On the other hand $l = k_1 + \frac{1}{2}$ and $k_3 = l + \frac{1}{2}$, hence $k_3 = k_1 + 1$. We now can say that since $k_1 + k_2 + k_3 = n$ then $k_1 + k_1 + k_1 + 1 = n$, therefore

$$k_1 = \frac{n-1}{3}.$$

We also know that $k_2 = k_1$ and $k_3 = k_1 + 1$, therefore

$$k_2 = \frac{n-1}{3} \quad \text{and} \quad k_3 = \frac{n+2}{3}.$$

This means that the point is

$$Q = \left(\frac{n-1}{3}, \frac{n-1}{3}, \frac{n+2}{3} \right). \quad \square$$

We can see that the term $\frac{n-1}{3}$ is not an integer unless $n = 4, 7, 10, \dots$, so we have to find the next closest point. Referring to Figure 4, let R be the closest integer point in the plane $k_1 + k_2 + k_3 = n$ to Q . Then by the Pythagorean Theorem,

$$QP^2 + QR^2 = PR^2,$$

because QP is perpendicular to the plane and QR lies in it. Therefore in order to minimize the distance PR we have to minimize QR , since QP is fixed.

To identify R we use a simple exchange argument.

Lemma 1. Let (k_1, k_2, k_3) be nonnegative integers with $k_1 + k_2 + k_3 = n$, and write $x_1 = k_1 + \frac{1}{2}$, $x_2 = k_2 + \frac{1}{2}$, $x_3 = k_3 - \frac{1}{2}$, so that $S = x_1^2 + x_2^2 + x_3^2$. If $x_i - x_j \geq 2$ for some $i \neq j$, then moving one coin from k_i to k_j gives another nonnegative triple with the same sum and strictly smaller S . Consequently, at an optimal point all the x_i differ from each other by at most 1.

Proof. The differences $x_i - x_j$ are integers. If $x_i - x_j \geq 2$ then $x_i \geq x_j + 2 \geq \frac{3}{2}$ (the smallest possible value of any x_j is $-\frac{1}{2}$), so $k_i \geq 1$ and the transfer keeps all coordinates nonnegative. The change in S is

$$(x_i - 1)^2 + (x_j + 1)^2 - x_i^2 - x_j^2 = 2 - 2(x_i - x_j) \leq -2 < 0. \quad \square$$

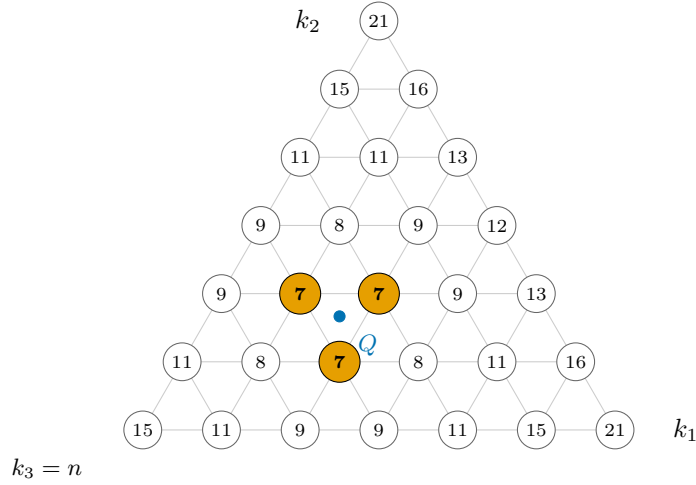


Figure 4: Closest point. The integer points of the plane $k_1 + k_2 + k_3 = 6$ with $k_i \geq 0$, each labelled with its value of t . The point $Q = (\frac{5}{3}, \frac{5}{3}, \frac{8}{3})$ lies inside the small triangle formed by the three optimal points, each with $t = 7$.

Theorem 4. *Let R be an optimal integer solution in the plane $k_1 + k_2 + k_3 = n$. Then the coordinates of R are given as follows:*

1. *Case 1: if $n = 3k$, for some integer k , then $R = (k, k, k)$ or $R = (k, k - 1, k + 1)$, or $R = (k - 1, k, k + 1)$;*
2. *Case 2: if $n = 3k + 1$, for some integer k , then $R = (k, k, k + 1)$;*
3. *Case 3: if $n = 3k + 2$, for some integer k , then $R = (k, k, k + 2)$ or $R = (k, k + 1, k + 1)$, or $R = (k + 1, k, k + 1)$.*

Proof. To prove this theorem we will consider three different cases. In each case, by Lemma 1 the three numbers x_1, x_2, x_3 of an optimal point differ by at most one, so they take at most two consecutive half-integer values, and their sum is $x_1 + x_2 + x_3 = n + \frac{1}{2}$.

Case 1: Let $n = 3k$, then

$$Q = \left(\frac{3k-1}{3}, \frac{3k-1}{3}, \frac{3k+2}{3} \right) = (k, k, k) + \left(-\frac{1}{3}, -\frac{1}{3}, \frac{2}{3} \right).$$

The point R with coordinates (k, k, k) belongs to the plane $k_1 + k_2 + k_3 = n$, since $n = 3k$. The distance QR is given by

$$QR^2 = \left(k - \left(k - \frac{1}{3} \right) \right)^2 + \left(k - \left(k - \frac{1}{3} \right) \right)^2 + \left(k - \left(k + \frac{2}{3} \right) \right)^2 = \frac{2}{3}.$$

We can also see that the point $R' = (k, k - 1, k + 1)$ belongs to the plane, since $k + (k - 1) + (k + 1) = 3k = n$, and the distance QR' is equal to

$$QR'^2 = \left(\frac{1}{3} \right)^2 + \left(-\frac{2}{3} \right)^2 + \left(\frac{1}{3} \right)^2 = \frac{2}{3}.$$

Lastly, by symmetry, the point $R'' = (k - 1, k, k + 1)$ is also in the plane and satisfies $QR''^2 = 2/3$. These are all the optimal points: here $x_1 + x_2 + x_3 = 3k + \frac{1}{2}$, which forces two of the x_i to equal

$k + \frac{1}{2}$ and one to equal $k - \frac{1}{2}$, and the three ways of choosing which one give exactly R , R' and R'' . Therefore all of these points provide optimal solutions to the problem.

Case 2: Let $n = 3k + 1$, then

$$Q = \left(\frac{(3k+1)-1}{3}, \frac{(3k+1)-1}{3}, \frac{(3k+1)+2}{3} \right) = (k, k, k+1).$$

In this case the point $Q = R$ with coordinates $(k, k, k+1)$ belongs to the plane, since $k + k + (k+1) = 3k + 1 = n$. Therefore the distance $QR = 0$, which makes the point R our optimal integer solution for the case $n = 3k + 1$, and it is the only one.

Case 3: Let $n = 3k + 2$, then

$$Q = \left(k + \frac{1}{3}, k + \frac{1}{3}, k + \frac{4}{3} \right) = (k, k, k+2) + \left(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3} \right).$$

The point R with coordinates $(k, k, k+2)$ belongs to the plane, since $n = 3k + 2$. The distance QR is given by

$$QR^2 = \left(-\frac{1}{3} \right)^2 + \left(-\frac{1}{3} \right)^2 + \left(\frac{2}{3} \right)^2 = \frac{2}{3}.$$

We can also see that the point $R' = (k, k+1, k+1)$ belongs to the plane, since $k + (k+1) + (k+1) = 3k + 2 = n$, and

$$QR'^2 = \left(-\frac{1}{3} \right)^2 + \left(\frac{2}{3} \right)^2 + \left(-\frac{1}{3} \right)^2 = \frac{2}{3}.$$

Lastly, by symmetry, the point $R'' = (k+1, k, k+1)$ is also in the plane and satisfies $QR''^2 = 2/3$. Similar to Case 1, these are all the optimal points: $x_1 + x_2 + x_3 = 3k + \frac{5}{2}$ forces two of the x_i to equal $k + \frac{1}{2}$ and one to equal $k + \frac{3}{2}$, giving exactly R , R' and R'' . Therefore all of these points provide optimal solutions to the problem. $\square$

By definition we knew that the base of the triangle created by the coins was n , from which we can see that the total amount of coins is equal to $\frac{n(n+1)}{2}$. This statement leads us to our last theorem. Here $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x .

Theorem 5. *Let the total amount of coins, T , be defined by*

$$T = \frac{n(n+1)}{2};$$

then

$$t = \left\lfloor \frac{n(n+1)}{6} \right\rfloor$$

represents the minimum amount of coins that need to be moved in order to turn the triangle upside down.

Proof. We have three different cases according to the residue of n modulo 3, and in each case we substitute the optimal points of Theorem 4 in

$$t = \frac{k_1(k_1+1)}{2} + \frac{k_2(k_2+1)}{2} + \frac{k_3(k_3-1)}{2}.$$

Case 1: Let $n = 3k$, so

$$T = \frac{3k(3k+1)}{2};$$

dividing by 3 on both sides we obtain

$$\frac{T}{3} = \frac{k(3k+1)}{2}.$$

If we plug in the coordinates (k, k, k) we get

$$t = \frac{k(k+1)}{2} + \frac{k(k+1)}{2} + \frac{k(k-1)}{2} = \frac{k(3k+1)}{2}.$$

Plugging in the point $(k, k-1, k+1)$ we get

$$t = \frac{k(k+1)}{2} + \frac{(k-1)k}{2} + \frac{(k+1)k}{2} = \frac{k(3k+1)}{2},$$

and the other point, $(k-1, k, k+1)$, gives the same value by symmetry. Therefore

$$t = \frac{k(3k+1)}{2} = \frac{T}{3} = \frac{n(n+1)}{6} \quad \text{when } n = 3k.$$

Case 2: Let $n = 3k+1$, so

$$T = \frac{(3k+1)(3k+2)}{2};$$

therefore

$$T - 1 = \frac{(3k+1)(3k+2)}{2} - 1 = \frac{9k(k+1)}{2}, \quad \text{so} \quad \frac{T}{3} = \frac{3k(k+1)}{2} + \frac{1}{3}.$$

Now we plug in $(k, k, k+1)$:

$$t = \frac{k(k+1)}{2} + \frac{k(k+1)}{2} + \frac{(k+1)k}{2} = \frac{3k(k+1)}{2}.$$

Hence $t = \frac{T}{3} - \frac{1}{3} = \left\lfloor \frac{T}{3} \right\rfloor$.

Case 3: Let $n = 3k+2$, so

$$T = \frac{(3k+2)(3k+3)}{2}, \quad \text{therefore} \quad \frac{T}{3} = \frac{(3k+2)(k+1)}{2}.$$

Substituting the coordinates $(k, k, k+2)$ in t we get

$$t = \frac{k(k+1)}{2} + \frac{k(k+1)}{2} + \frac{(k+2)(k+1)}{2} = \frac{3k^2 + 5k + 2}{2} = \frac{(3k+2)(k+1)}{2},$$

hence $\frac{T}{3} = t$. The coordinates $(k, k+1, k+1)$ give

$$t = \frac{k(k+1)}{2} + \frac{(k+1)(k+2)}{2} + \frac{(k+1)k}{2} = \frac{3k^2 + 5k + 2}{2},$$

and $(k+1, k, k+1)$ gives the same by symmetry; hence $\frac{T}{3} = t$ in this case as well.

By proving these three cases we can conclude that the least amount of moves is equal to

$$t = \left\lfloor \frac{T}{3} \right\rfloor = \left\lfloor \frac{n(n+1)}{6} \right\rfloor. \quad \square$$

After solving these five theorems we are now able to say that the problem can be solved by knowing the amount of coins located in the base of the triangle, which is actually the same as any side. The formula that can be used to solve the problem is

$$t = \left\lfloor \frac{n(n+1)}{6} \right\rfloor.$$

4 Three dimensions and beyond

The coin problem has a natural analogue in any dimension $d \geq 1$: a pile of spheres in the shape of a d -simplex with n spheres on each edge, to be turned into its point reflection with the fewest moves. For $d = 3$ this is the familiar tetrahedral pile of cannonballs. In this section we show that the reduction to a nearest-point problem survives in every dimension, we obtain the closed form for $d = 3$, and we report what changes for $d \geq 4$.

4.1 Setting

Let $x = (x_1, \dots, x_d) \in \mathbb{Z}^d$ and write $|x| = x_1 + \dots + x_d$. The pile is

$$U = \{x \in \mathbb{Z}^d : x_i \geq 0, |x| \leq n-1\}, \quad |U| = T_n^{(d)} = \binom{n+d-1}{d},$$

and its point reflection translated by $v \in \mathbb{Z}^d$ is

$$D(v) = \{x \in \mathbb{Z}^d : x_i \leq v_i \text{ for all } i, |x| \geq |v| - (n-1)\}.$$

For $d = 2$ these are the sets U and $D(p, q)$ of Section 2. Exactly as in the proof of Theorem 1, the least number of moves that takes U to $D(v)$ is $|U \setminus D(v)| = T_n^{(d)} - |U \cap D(v)|$, so we must maximize the overlap

$$\Omega(v) = |U \cap D(v)| = \#\{x \in \mathbb{Z}^d : 0 \leq x_i \leq v_i, L \leq |x| \leq n-1\}, \quad L = |v| - (n-1). \quad (2)$$

If some $v_i < 0$ the overlap is empty, so we take $v_i \geq 0$ throughout. We call v *balanced* if $\max_i v_i - \min_i v_i \leq 1$; for each $w \geq 0$ there is exactly one balanced v with $|v| = w$ up to permutation of coordinates, and we denote it $v(w)$.

4.2 Two lemmas

For $v \in \mathbb{Z}_{\geq 0}^d$ and $s \geq 0$ let $N_v(s)$ be the number of x with $0 \leq x_i \leq v_i$ and $|x| = s$. This is the coefficient of q^s in

$$\prod_{i=1}^d (1 + q + \dots + q^{v_i}),$$

and by (2), $\Omega(v) = \sum_{s=\max(L,0)}^{n-1} N_v(s)$.

Lemma 2 (Balancing). *Let $v \in \mathbb{Z}_{\geq 0}^d$ with $v_i \geq v_j + 2$ for some $i \neq j$, and let $v' = v - e_i + e_j$. Then $\Omega(v') \geq \Omega(v)$. Consequently there is an optimal placement that is balanced.*

Proof. Since $|v'| = |v|$, the range of summation in (2) is the same for v and v' , and it suffices to show $N_{v'}(s) \geq N_v(s)$ for every s . Write the product above as $F \cdot G$, where F is the product of the factors for i and j and G is the product of the others. Put $a = v_i$, $b = v_j$, so $a \geq b + 2$. The coefficient of q^s in $F = (1 + \dots + q^a)(1 + \dots + q^b)$ counts pairs (x_i, x_j) with $0 \leq x_i \leq a$, $0 \leq x_j \leq b$, $x_i + x_j = s$, and for $0 \leq s \leq a + b$ this number is

$$c_s = \min(s, b, a + b - s) + 1.$$

For v' the pair (a, b) becomes $(a-1, b+1)$, which still satisfies $a-1 \geq b+1$, so the corresponding coefficient is $c'_s = \min(s, b+1, a+b-s) + 1 \geq c_s$. Thus $F' \geq F$ coefficient by coefficient, and

since G has nonnegative coefficients, $F'G \geq FG$ coefficient by coefficient. Repeating the transfer while some two coordinates differ by at least 2 terminates, because $\sum_i v_i^2$ strictly decreases at each step, and ends at a balanced vector. $\square$

Lemma 3 (One more sphere in the box). *Let $v \in \mathbb{Z}_{\geq 0}^d$, fix i , and let $v' = v + e_i$. Then*

$$\Omega(v') - \Omega(v) \begin{cases} \geq 0 & \text{if } |v| + v_i \leq 2n - 3, \\ \leq 0 & \text{if } |v| + v_i \geq 2n - 2. \end{cases}$$

Proof. Write y for the vector of the $d - 1$ coordinates other than x_i , ranging over the box $B' = \{0 \leq y_j \leq v_j\}$, and let $L = |v| - (n - 1)$, so that $L' = L + 1$ is the lower limit for v' . The box for v' is the box for v together with the layer $x_i = v_i + 1$. Hence

$$\Omega(v') = \left[\Omega(v) - N_v(L) \right] + \#\{y \in B' : L + 1 \leq |y| + v_i + 1 \leq n - 1\},$$

where the bracket removes the points with $|x| = L$, which are no longer in the slab, and the last term counts the new layer. Also $N_v(L) = \sum_{x_i=0}^{v_i} \#\{y \in B' : |y| = L - x_i\} = \#\{y \in B' : L - v_i \leq |y| \leq L\}$. Therefore

$$\Omega(v') - \Omega(v) = \#\{y \in B' : L - v_i \leq |y| \leq n - 2 - v_i\} - \#\{y \in B' : L - v_i \leq |y| \leq L\}.$$

If $n - 2 - v_i \geq L$, the difference equals $\#\{y \in B' : L + 1 \leq |y| \leq n - 2 - v_i\} \geq 0$; if $n - 2 - v_i \leq L - 1$, it equals $-\#\{y \in B' : n - 1 - v_i \leq |y| \leq L\} \leq 0$. Since $L = |v| - n + 1$, the first condition is $|v| + v_i \leq 2n - 3$ and the second is $|v| + v_i \geq 2n - 2$. $\square$

4.3 The optimal placement in every dimension

Theorem 6. *For every $d \geq 1$ and $n \geq 1$, the maximum overlap is attained at the balanced vector $v(w^*)$, where*

$$w^* = \min \left\{ w \geq 0 : w + \left\lfloor \frac{w}{d} \right\rfloor \geq 2n - 2 \right\}.$$

Hence the least number of moves is $t_{\min}^{(d)}(n) = T_n^{(d)} - \Omega(v(w^*))$.

Proof. By Lemma 2 it suffices to maximize over balanced vectors, that is over the sequence $v(0), v(1), v(2), \dots$. Passing from $v(w)$ to $v(w + 1)$ adds 1 to a coordinate equal to $\lfloor w/d \rfloor$, so by Lemma 3 the overlap does not decrease while $w + \lfloor w/d \rfloor \leq 2n - 3$ and does not increase once $w + \lfloor w/d \rfloor \geq 2n - 2$. The function $w \mapsto w + \lfloor w/d \rfloor$ is nondecreasing, so $\Omega(v(w))$ is nondecreasing up to w^* and nonincreasing afterwards, and its maximum is attained at w^* . (Ties are possible on either side of w^* when the counts in Lemma 3 vanish.) $\square$

Remark 1. For $d = 2$ and $n = 3k + 1$ the theorem gives $w^* = 4k$ and $v(w^*) = (2k, 2k)$, which is the placement $(p, q) = (n - 1 - k, n - 1 - k)$ corresponding to the unique optimal triple $(k, k, k + 1)$ of Theorem 4; the other residue classes agree in the same way. So Theorem 6 is a second route to the coin result, although it says nothing about the value of t until Ω is evaluated. In two and three dimensions the evaluation is easy because the uncovered spheres form disjoint corner simplices; in higher dimensions they do not.

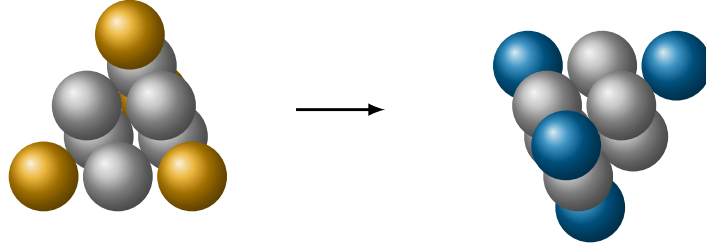


Figure 5: A tetrahedral pile with $n = 3$ (10 spheres). The optimal placement moves exactly the four corner spheres (orange) to the four new positions (blue); the central octahedron of six spheres stays in place.

4.4 Tetrahedral piles

Theorem 7. Let $T_n^{(3)} = \frac{n(n+1)(n+2)}{6}$ be the number of spheres in a tetrahedral pile with n on each edge. The least number of spheres that must be moved to turn the pile into its point reflection is

$$t_{\min}^{(3)}(n) = \begin{cases} \frac{2m(m+1)(m+2)}{3}, & n = 2m + 1, \\ \frac{T_n^{(3)}}{2} = \frac{m(m+1)(2m+1)}{3}, & n = 2m. \end{cases}$$

Proof. By Theorem 6 with $d = 3$: if $n = 2m + 1$ then $w = 3m$ gives $w + \lfloor w/3 \rfloor = 4m = 2n - 2$ while $w = 3m - 1$ gives $4m - 2$, so $w^* = 3m$ and $v = (m, m, m)$; if $n = 2m$ then $w = 3m - 1$ gives $4m - 2 = 2n - 2$ while $w = 3m - 2$ gives $4m - 3$, so $w^* = 3m - 1$ and $v = (m, m, m - 1)$.

Now count $U \setminus D(v)$ as in Theorem 1. A sphere $x \in U$ is uncovered when $x_1 \geq v_1 + 1$, or $x_2 \geq v_2 + 1$, or $x_3 \geq v_3 + 1$, or $|x| \leq L - 1$ with $L = |v| - n + 1$. These four sets are corner tetrahedra of U with edge lengths

$$s_1 = n - 1 - v_1, \quad s_2 = n - 1 - v_2, \quad s_3 = n - 1 - v_3, \quad s_4 = L,$$

containing $\binom{s_i+2}{3}$ spheres each (the substitution $x_1 = x'_1 + v_1 + 1$ turns the first set into $\{x'_1, x_2, x_3 \geq 0, x'_1 + x_2 + x_3 \leq s_1 - 1\}$, and similarly for the others). Two of these corners intersect only if the sum of their edge lengths is at least $n + 1$: for the first two, a common point needs $v_1 + v_2 + 2 \leq |x| \leq n - 1$, that is $s_1 + s_2 \geq n + 1$; for the first and the fourth, it needs $v_1 + 1 \leq x_1 \leq |x| \leq L - 1$, that is $s_1 + s_4 = |v| - v_1 \geq n + 1$; the other pairs are symmetric.

If $n = 2m + 1$ then $(s_1, s_2, s_3, s_4) = (m, m, m, m)$ and every pair sums to $2m = n - 1$; if $n = 2m$ then $(s_1, s_2, s_3, s_4) = (m - 1, m - 1, m, m)$ and every pair sums to at most $2m = n$. In both cases the corners are pairwise disjoint and

$$t_{\min}^{(3)}(n) = \sum_{i=1}^4 \binom{s_i+2}{3} = \begin{cases} 4\binom{m+2}{3} = \frac{2m(m+1)(m+2)}{3}, & n = 2m + 1, \\ 2\binom{m+1}{3} + 2\binom{m+2}{3} = \frac{m(m+1)(2m+1)}{3}, & n = 2m, \end{cases}$$

and for $n = 2m$ the last expression equals $\frac{2m(2m+1)(2m+2)}{12} = T_n^{(3)}/2$. $\square$

The values for $n \leq 10$ are

n	1	2	3	4	5	6	7	8	9	10
$T_n^{(3)}$	1	4	10	20	35	56	84	120	165	220
$t_{\min}^{(3)}(n)$	0	2	4	10	16	28	40	60	80	110

The sequence $t_{\min}^{(3)}(n)$ is OEIS A006584 shifted by one index [5]; that entry lists it as $m(m^2-4)/12$ or $m(m^2-1)/12$ according to the parity of $m = n+1$, which agrees with the theorem, but it carries no interpretation in terms of piles of spheres. Exactly half of the spheres move when n is even, and the fraction tends to $\frac{1}{2}$ as $n \rightarrow \infty$, compared with $\frac{1}{3}$ for coins. In the continuum the point reflection of a regular tetrahedron through its centroid overlaps it in an octahedron of half the volume, and the point reflection of a triangle overlaps it in a hexagon of two thirds the area, which is the same statement.

4.5 Higher dimensions

Theorem 6 identifies the optimal placement in every dimension, so $t_{\min}^{(d)}(n)$ can be computed from (2) for any d and n . The first values are

n	1	2	3	4	5	6	7	8	9	10	11	12
$t_{\min}^{(4)}(n)$	0	3	9	21	44	75	132	206	309	451	620	861
$t_{\min}^{(5)}(n)$	0	4	15	36	90	182	321	572				

Neither sequence appears in the OEIS at the time of writing.

What does not survive beyond $d = 3$ is the corner-counting step. In dimension d the uncovered spheres form $d+1$ corner simplices whose edge lengths sum to $(d-1)(n-1)$, and if those corners were disjoint at the optimum the answer would be the balanced sum $\sum_{i=1}^{d+1} \binom{s_i+d-1}{d}$ with the s_i as equal as possible. For $d = 2$ and $d = 3$ this is what Theorems 5 and 7 say. For $d = 4$, $n = 5$ the balanced sum is 45 but the true minimum is 44; for $d = 4$, $n = 7$ the values are 135 against 132; for $d = 5$, $n = 5$ they are 96 against 90. The optimal placement is still balanced, as Theorem 6 guarantees, but the corner simplices now overlap, and the overcount in the balanced sum is exactly the inclusion–exclusion correction. A closed form for $d \geq 4$ would have to account for these overlaps and we leave it open.

Remark 2 (Verification). All statements above were checked by exhaustive computation independent of the proofs: for $d = 2$ the formula $\lfloor n(n+1)/6 \rfloor$ and the count of optimal placements (1 when $n \equiv 1 \pmod{3}$, else 3) for $n \leq 79$; for $d = 3$ the closed form of Theorem 7 for $n \leq 40$; the monotonicity pattern of Lemma 3 along the balanced chain and the equality of its maximum with the global maximum for $d \leq 5$, $n \leq 15$; and Lemma 2 on every transfer for $d \leq 4$, $n \leq 10$. The code is short and is available from the author.

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